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Gauge invariant formulation of $N = 2$ Toda and KdV systems in extended superspace

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Abstract. We give a gauge invariant formulation of $N = 2$ supersymmetric abelian Toda field equations in $N = 2$ superspace. Superconformal invariance is studied. The conserved currents are shown to be associated with Drinfeld–Sokolov type gauges. The extension to non-abelian $N = 2$ Toda equations is discussed. Very similar methods are then applied to a matrix formulation in $N = 2$ superspace of one of the $N = 2$ KdV hierarchies.

0. Introduction

The $N = 2$ supersymmetric Liouville equation together with its Lax representation in superspace was first given in [6]. The generalization to the abelian Toda equations have been derived and studied in [2]. These models are associated with the principal embedding of the superalgebra $sl(2|1)$ inside $sl(n+1|n)$. Here we construct a set of gauge invariant equations in $N = 2$ superspace which, in a particular gauge, reduce to the $N = 2$ supersymmetric abelian Toda equations. This is in the spirit of the works [3] and [4], where such a gauge invariant formulation was constructed for bosonic Toda models in ordinary space, and for supersymmetric Toda models in $N = 1$ superspace. The $N = 1$ superspace approach leads to a natural interpertation for the use of superalgebras admitting a principal $osp(1|2)$ embedding [5]. The $N = 2$ abelian Toda equations correspond to the cases when this principal $osp(1|2)$ embedding may be extended to an $sl(2|1)$ embedding. The consistency of the gauge invariant equations is ensured by an interesting interplay between the geometry of the extended superspace and the structure of the $sl(2|1)$ superalgebra. Such a gauge invariant formulation allows for an easy discussion of the conserved currents of the Toda equations. However, it should be stressed that the formulation we give in this article is restricted to the discussion of field equations. We do not have an explicitly supersymmetric action. A Hamiltonian formulation in extended superspace is presently out of reach†. Another difference with previous works is that in the $N = 2$ superspace, it seems hard to interpret the gauge invariant field equations as a gauged WZNW model.

The paper is organized as follows. In the first section we recall some basic facts about the superalgebra $sl(n+1|n)$ and its principal $sl(2|1)$ embedding. In section 2, we write down the gauge invariant field equations and establish the relation with $N = 2$ Toda equations. Then we discuss the conserved currents of $N = 2$ Toda equations and their superconformal transformations. Finally we apply this formalism to non-abelian $N = 2$ Toda equations. In section 3, we construct in $N = 2$ superspace a matrix Lax formulation of an $N = 2$ supersymmetric KdV hierarchy.

† It is to be noted that a Hamiltonian reduction approach to $N = 2\mathcal{W}$ algebras was developed in [6].

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1. The superalgebra $sl(n+1|n)$ and its principal $sl(2|1)$ subalgebra

This section is devoted to a short introduction to the superalgebras $sl(n+1|n)$. We shall choose a basis which allows for an easy description of the principal $sl(2|1)$ subalgebra.

We consider the set of $(2n+1)\times(2n+1)$ real matrices. A convenient basis is given by the matrices $E_{i,j}$ such that $(E_{i,j})_{kl} = \delta_{ik}\delta_{jl}$. We define the supertrace of a matrix M by the alternate sum

$$\mathrm{str}\, M = \sum_{i=1}^{2n+1} (-1)^{i+1} M_{ii}. \tag{1.1}$$

The supercommutator of two matrices M and N is

$$[M, N\}_{ij} = \sum_{k=1}^{2n+1} (M_{ik}N_{kj} - (-1)^{(i+k)(k+j)} N_{ik}M_{kj}). \tag{1.2}$$

One then easily checks that $\mathrm{str}[M, N\} = 0$. The superalgebra $\mathcal{G} = sl(n+1|n)$ is the set of matrices with zero supertrace. A Cartan subalgebra of $\mathcal{G}$ is generated by the diagonal matrices

$$H_i = E_{i,i} + E_{i+1,i+1} \qquad 1 \leqslant i \leqslant 2n. \tag{1.3}$$

The superalgebra $\mathcal{G}$ is Z_2 graded, $\mathcal{G} = \mathcal{G}_{\bar{0}} + \mathcal{G}_{\bar{1}}$. The Z_2 -grading of a matrix $E_{i,j}$ is $i+j$ modulo 2. We shall denote by a hat the superalgebra automorphism which reverses the sign of odd elements,

$$(\hat{M})_{ij} = (-1)^{i+j} M_{ij}. \tag{1.4}$$

The matrices $E_{i,i+1}$ just above the diagonal are associated with a complete set of fermionic simple roots. We now define the principal $sl(2|1)$ embedding. The Cartan subalgebra is spanned by

$$H = \sum_{i=1}^{n} \frac{n-i+1}{2} H_{2i-1} \qquad \bar{H} = -\sum_{i=1}^{n} \frac{i}{2} H_{2i}. \tag{1.5}$$

The matrices μ_+, $\bar{\mu}_+$ associated with positive simple fermionic roots are

$$\mu_+ = \sum_{i=1}^{n} E_{2i-1,2i} \qquad \bar{\mu}_+ = \sum_{i=1}^{n} E_{2i,2i+1} \tag{1.6}$$

and the matrices μ_-, $\bar{\mu}_-$ associated with negative simple fermionic roots are

$$\mu_- = \sum_{i=1}^{n} (n-i+1) E_{2i,2i-1} \qquad \bar{\mu}_- = -\sum_{i=1}^{n} i\, E_{2i+1,2i} \tag{1.7}$$

The non-zero super-commutators are

$$[\bar{H}, \mu_\pm] = \pm\tfrac{1}{2}\mu_\pm \qquad [H, \bar{\mu}_\pm] = \pm\tfrac{1}{2}\bar{\mu}_\pm \tag{1.8}$$

$$\{\mu_+, \mu_-\} = 2H \qquad \{\bar{\mu}_+, \bar{\mu}_-\} = 2\bar{H} \tag{1.9}$$

$$\{\mu_+, \bar{\mu}_+\} = E_{++} \qquad \{\mu_-, \bar{\mu}_-\} = E_{--} \tag{1.10}$$

where E_{++} and E_{--} are associated with the bosonic roots of $sl(2|1)$. This algebra contains the principal $osp(2|1)$ subalgebra of $\mathcal{G}$. The Cartan generator is $e_0 = H + \bar{H}$, the operators associated with positive and negative fermionic simple roots are $f_+ = \mu_+ + \bar{\mu}_+$, $f_- = \mu_- + \bar{\mu}_-$. The operator $\mathrm{ad}e_0$ is diagonalizable. The eigenvalues are $i/2$, $-n \leqslant i \leqslant n$. We denote the corresponding eigenspaces by $\mathcal{G}_{i/2}$. This defines a half-integer grading of $\mathcal{G}$, $\mathcal{G} = \oplus_{i=-n}^{n} \mathcal{G}_{i/2}$. In the basis that we have chosen, positive (negative) eigenvalues correspond to upper (lower) triangular matrices and $\mathcal{G}_0$ is the Cartan subalgebra. Even elements have integer gradation, and odd elements have half-integer gradation. In particular, μ_+ and $\bar{\mu}_+$ (μ_- and $\bar{\mu}_-$) belong to the eigenspace $\mathcal{G}_{\frac{1}{2}}$ ($\mathcal{G}_{-\frac{1}{2}}$).

Finally, in the following we always use matrices taking values in some Grassmann algebra $\mathcal{G}r = \mathcal{G}r_{\bar{0}} \oplus \mathcal{G}r_{\bar{1}}$. These matrices will be called even if they belong to $\mathcal{E} = \mathcal{G}_{\bar{0}} \otimes \mathcal{G}r_{\bar{0}} \oplus \mathcal{G}_{\bar{1}} \otimes \mathcal{G}r_{\bar{1}}$ and odd if they belong to $\mathcal{O} = \mathcal{G}_{\bar{0}} \otimes \mathcal{G}r_{\bar{1}} \oplus \mathcal{G}_{\bar{1}} \otimes \mathcal{G}r_{\bar{0}}$. We then define the supercommutator of two matrices A and B belonging to $\mathcal{E}$ or $\mathcal{O}$ by

$$[A, B\} = \sum_{i,j,k,l} A_{ij} B_{kl} [E_{i,j}, E_{k,l}\}. \tag{1.11}$$

In the various cases, this takes the matrix product forms listed in table 1.

Table 1.

$[A, B\}$	$A \in \mathcal{E}$	$A \in \mathcal{O}$
$B \in \mathcal{E}$	$AB - BA$	$AB - \hat{B}A$
$B \in \mathcal{O}$	$AB - B\hat{A}$	$AB + \hat{B}\hat{A}$

2. Gauge invariant formulation of the Toda field equations

The method used in this section is an extension of that developed in [3]. However, our method only applies to the field equations, and we do not have an explicitly supersymmetric action or Hamiltonian formulation.

2.1. Zero curvature equations and constraints

We denote the supergroup corresponding to the superalgebra $\mathcal{G} = sl(n+1|n)$ by $G = SL(n+1|n)$. We use the grading described in the last section, and separate $\mathcal{G}$ as $\mathcal{G} = \mathcal{G}_{<0} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{>0}$. $\mathcal{G}_{>0}$ contains all upper triangular matrices, $\mathcal{G}_{<0}$ all lower triangular matrices. $\mathcal{G}_0$ is the Cartan subalgebra spanned by diagonal matrices. The supergroups corresponding respectively to $\mathcal{G}_{<0}$, $\mathcal{G}_0$, $\mathcal{G}_{>0}$ will be denoted by $G_{<0}$, G_0, $G_{>0}$. The elements of $G_{>0}$ ($G_{<0}$) are upper (lower) triangular even matrices with ones on the diagonal. The coordinates of the $N = 2$ superspace are the light-cone coordinates (x^{++}, x^{--}) together with the Grassmann coordinates $(\theta^+, \bar{\theta}^+, \theta^-, \bar{\theta}^-)$. Notice that we use an $N = 2$ supersymmetry algebra with $SO(1, 1)$ automorphism group. The covariant spinor derivatives are

$$D_+ = \frac{\partial}{\partial\theta^+} + \bar{\theta}^+\partial_{++} \qquad \bar{D}_+ = \frac{\partial}{\partial\bar{\theta}^+} + \theta^+\partial_{++} \qquad \partial_{++} = \frac{\partial}{\partial x^{++}} \tag{2.1}$$

and D_-, $\bar{D}_-$ are similarly defined. These derivatives satisfy the algebra

$$D_+^2 = \bar{D}_+^2 = 0 \qquad \{D_+, \bar{D}_+\} = 2\partial_{++} \qquad \{\tilde{D}_+, \tilde{D}_-\} = 0 \tag{2.2}$$

where we use a tilde as a generic notation for unbarred or barred objects, $\tilde{D}_+ = D_+$ or $\bar{D}_+$.

We now introduce the superfields that we need for the gauge invariant formulation of Toda equations. The first is $g(x, \theta, \bar{\theta})$ which takes values in the supergroup G. Then there are spinor gauge superfields $A_\pm(x, \theta, \bar{\theta})$, $\bar{A}_\pm(x, \theta, \bar{\theta})$. A_+ and $\bar{A}_+$ are upper triangular odd matrices, A_- and $\bar{A}_-$ are lower triangular odd matrices. The gauge transformations

$R(x,\theta,\bar\theta)\in G_{>0}$ and $L(x,\theta,\bar\theta)\in G_{<0}$ act on the superfield $g(x,\theta,\bar\theta)$ by left and right translations

$$g\longrightarrow LgR \tag{2.3}$$

and on the gauge superfields by

$$\tilde A_+\longrightarrow \hat R^{-1}\tilde D_+R+\hat R^{-1}\tilde A_+R \tag{2.4}$$
$$\tilde A_-\longrightarrow \tilde D_-LL^{-1}+\hat L\tilde A_-L^{-1}. \tag{2.5}$$

At that point, the connections $\tilde A_+$ and $\tilde A_-$ transform under different gauge groups. However, the field g acts as a 'bridge' between the two gauge groups and may be used to construct connections transforming under the same gauge group,

$$\tilde B_+=-\hat g\tilde A_+g^{-1}+\tilde D_+gg^{-1}\qquad \tilde B_-=\tilde A_-. \tag{2.6}$$

The connections $\tilde B_\pm$ only transform under the left gauge group

$$\tilde B_\pm\longrightarrow(\tilde D_\pm L)L^{-1}+\hat L\tilde B_\pm L^{-1}. \tag{2.7}$$

One could equivalently use a set of connections transforming only under the right group

$$\tilde C_+=\tilde A_+\qquad \tilde C_-=-\hat g^{-1}\tilde A_-g+\hat g^{-1}\tilde D_-g \tag{2.8}$$
$$\tilde C_\pm\longrightarrow\hat R^{-1}\tilde D_\pm R+\hat R^{-1}\tilde C_\pm R. \tag{2.9}$$

We now require the $\tilde B_\pm$ connections to have zero curvature. This gives ten equations corresponding to the vanishing of the spinor–spinor components of the curvature. Two of these equations simply determine the vector component of the connection in terms of the spinor components

$$2B_{++}=D_+\bar B_++\bar D_+B_+-[\hat B_+,\bar B_+\} \tag{2.10}$$

and there is an analogous equation determining B_{--}. These are just the familiar conventional constraints of super-Yang–Mills theories. Among the eight remaining equations, four involve only one light-cone chirality

$$D_+B_+-\hat B_+B_+=0\Leftrightarrow D_+A_++\hat A_+A_+=0 \tag{2.11}$$
$$D_-B_--\hat B_-B_-=0\Leftrightarrow D_-A_--\hat A_-A_-=0 \tag{2.12}$$

and there are similar equations for $\bar B_+$, $\bar B_-$, or, equivalently, for $\bar A_+$, $\bar A_-$. These again are well known constraints of super-Yang–Mills theories, usually referred to as representation preserving constraints. Finally, there remains four equations involving both light-cone chiralities

$$\tilde D_+\tilde B_-+\tilde D_-\tilde B_+-[\hat{\tilde B}_+,\tilde B_-\}=0. \tag{2.13}$$

In order to recover the Toda equations in this framework, one has to add to the zero curvature equations some gauge invariant constraints. These constraints involve the roots of the principal $sl(2)$ embedding in the bosonic case, and the simple fermionic roots of the principal $osp(2|1)$ embedding in the $N=1$ supersymmetric case. Here it involves the simple fermionic roots of the principal $sl(2|1)$ embedding, e.g.

$$(\tilde B_+)_{>0}=\tilde\mu_+\qquad (\tilde C_-)_{<0}=\tilde\mu_-. \tag{2.14}$$

The gauge invariance of these constraints as usual follows from the fact that μ_+ and $\bar\mu_+$ (μ_- and $\bar\mu_-$) belong to the space $\mathcal G_{\frac12}$ ($\mathcal G_{-\frac12}$) with smallest positive (negative) grade. Before establishing the relation with the Toda equations as given in [2], let us study the superconformal invariance of our equations. Beside gauge invariance, let us note that the zero curvature equations possess two additional symmetries. The first is a loop invariance, where the infinitesimal parameters l_0 and r_0 belong to the grade zero subalgebra $\mathcal{G}_0$. It acts on the fields by

$$\delta g = l_0 g + g r_0 \qquad \delta \tilde{A}_+ = [\tilde{A}_+, r_0] \qquad \delta \tilde{A}_- = [l_0, \tilde{A}_-]. \tag{2.15}$$

The parameters depend only on one light-cone chirality,

$$D_- l_0 = \bar{D}_- l_0 = 0 \qquad D_+ r_0 = \bar{D}_+ r_0 = 0. \tag{2.16}$$

The second symmetry is $N = 2$ superconformal invariance. Let us denote by $K(\epsilon^{++}, \epsilon^{--})$ the differential operator

$$\begin{aligned} K(\epsilon^{++}, \epsilon^{--}) = {} & \epsilon^{++}\partial_{++} + \tfrac{1}{2} D_+ \epsilon^{++} \bar{D}_+ + \tfrac{1}{2} \bar{D}_+ \epsilon^{++} D_+ + \epsilon^{--}\partial_{--} + \tfrac{1}{2} D_- \epsilon^{--} \bar{D}_- \\ & + \tfrac{1}{2} \bar{D}_- \epsilon^{--} D_- \end{aligned} \tag{2.17}$$

where the infinitesimal parameters $\epsilon^{\pm\pm}$ depend only on one light-cone chirality,

$$D_- \epsilon^{++} = \bar{D}_- \epsilon^{++} = 0 \qquad D_+ \epsilon^{--} = \bar{D}_+ \epsilon^{--} = 0. \tag{2.18}$$

The superconformal transformations of the scalar superfield g and of the spinor components of the connection are

$$\begin{aligned} \delta g &= K(\epsilon^{++}, \epsilon^{--}) g \\ \delta A_\pm &= K(\epsilon^{++}, \epsilon^{--}) A_\pm + \tfrac{1}{2} D_\pm \bar{D}_\pm \epsilon^{\pm\pm} A_\pm \\ \delta \bar{A}_\pm &= K(\epsilon^{++}, \epsilon^{--}) \bar{A}_\pm + \tfrac{1}{2} \bar{D}_\pm D_\pm \epsilon^{\pm\pm} \bar{A}_\pm. \end{aligned} \tag{2.19}$$

The constraints (2.14) are not invariant separately under Kac–Moody and superconformal transformations. There is, however, a unique choice of the Kac–Moody parameters l_0 and r_0 in terms of the superconformal parameters ϵ^{++} and ϵ^{--} such that the combined transformations leave the constraints invariant. It is given by

$$l_0 = -D_+ \bar{D}_+ \epsilon^{++} \bar{H} - \bar{D}_+ D_+ \epsilon^{++} H \qquad r_0 = -D_- \bar{D}_- \epsilon^{--} \bar{H} - \bar{D}_- D_- \epsilon^{--} H. \tag{2.20}$$

Thus we conclude that the constraints (2.14) do allow for $N = 2$ superconformal invariance, provided the superconformal transforation laws are suitably modified.

2.2. *Relation with $N = 2$ Toda equations*

This relation is most easily obtained by choosing the particular gauge where the superfield g is constrained to belong to the subgroup G_0,

$$g = g_0 = \exp\left(\sum_{i=1}^{2n} \phi^i H_i \right). \tag{2.21}$$

In this gauge, the constraints (2.14) simply determine the spinor connections

$$\tilde{A}_+ = -g_0^{-1} \tilde{\mu}_+ g_0 \qquad \tilde{A}_- = -g_0 \tilde{\mu}_- g_0^{-1}. \tag{2.22}$$

We then have to take into account the fact that these connections are constrained superfields. They satisfy equations (2.11) and (2.12), which may be translated into the following constraints on g_0

$$\begin{aligned} &[\mu_+, D_+ g_0 g_0{}^{-1}] = 0 \qquad [\bar{\mu}_+, \bar{D}_+ g_0 g_0{}^{-1}] = 0 \\ &[\mu_-, g_0^{-1} D_- g_0] = 0 \qquad [\bar{\mu}_-, g_0{}^{-1} \bar{D}_- g_0] = 0. \end{aligned} \tag{2.23}$$

More explicitly, the fields ϕ^i satisfy the following supersymmetric chirality constraints

$$\bar{D}_+ \phi^{2i-1} = \bar{D}_- \phi^{2i-1} = 0 \qquad D_+ \phi^{2i} = D_- \phi^{2i} = 0. \tag{2.24}$$

Among the four zero curvature equations (2.13), there remains only two dynamical equations

$$D_-(D_+g_0g_0{}^{-1}) + \{\mu_+, g_0\mu_-g_0{}^{-1}\} = 0 \qquad \bar{D}_-(\bar{D}_+g_0g_0{}^{-1}) + \{\bar{\mu}_+, g_0\bar{\mu}_-g_0{}^{-1}\} = 0 \quad (2.25)$$

which, when expressed on the fields ϕ^i, become the $N = 2$ Toda equations given in [2],

$$D_+D_-\phi^{2j-1} = (n-j+1)\,\mathrm{e}^{\phi^{2j}-\phi^{2j-2}} \qquad \bar{D}_+\bar{D}_-\phi^{2j} = -j\,\mathrm{e}^{\phi^{2j+1}-\phi^{2j-1}}. \quad (2.26)$$

The next section will be devoted to a discussion of conserved currents and of various Drinfeld–Sokolov type gauges.

2.3. Conserved currents and gauge choices

From the zero curvature equations it is clear that any gauge invariant function $F(B_+, \bar{B}_+)$ or $G(C_-, \bar{C}_-)$ will satisfy the light-cone chirality conditions

$$D_-F(B_+, \bar{B}_+) = \bar{D}_-F(B_+, \bar{B}_+) = 0 \qquad D_+G(C_-, \bar{C}_-) = \bar{D}_+G(C_-, \bar{C}_-) = 0. \quad (2.27)$$

Let us concentrate on the functions $F(B_+, \bar{B}_+)$. We shall now find a generating set for these gauge invariant functions [8]. The elements of this set are differential polynomials in the matrix elements of $(B_+)_{\leqslant 0}$, $(\bar{B}_+)_{\leqslant 0}$. These polynomials will be found by exhibiting a unique gauge transformation which brings the connections to a prescribed form. In order to do this, we use our knowledge of the $N = 1$ case [4]. We first consider the sum $B_+ + \bar{B}_+$, which is constrained by equation (2.14) to satisfy $(B_+ + \bar{B}_+)_{>0} = f_+$. The adjoint action of the $osp(2|1)$ generator f_+ on the gauge algebra $\mathcal{G}_-$ is non-degenerate. Let us separate $\mathcal{G}_{\leqslant 0}$ into $(\mathrm{Im\ ad}\, f_+)_{\leqslant 0}$ plus some graded supplementary subspace $V_{\leqslant 0}$. This space has one basis element $T_{-i/2}$ at each strictly negative grade. Then there exist a unique gauge transformation $\mathrm{e}^F \in G_{<0}$ such that

$$\mathrm{e}^{\hat{F}}(D_+ + \bar{D}_+ + (B_+)_{\leqslant 0} + (\bar{B}_+)_{\leqslant 0} + f_+)\,\mathrm{e}^{-F} = D_+ + \bar{D}_+ + W + f_+ \quad (2.28)$$

and $W = \sum_{i=1}^{2n} W_{-i/2}T_{-i/2}$ belongs to $V_{\leqslant 0}$. Moreover, both F and W are differential polynomials in the matrix elements of $(B_+)_{\leqslant 0}$ and $(\bar{B}_+)_{\leqslant 0}$. It remains to be seen whether all polynomials $W_{-i/2}$ are independent or not. To do this, we restrict to the special case when the basis element of $V_{\leqslant 0}$ at half-integer grade $T_{-i+1/2}$ are related to the basis element at integer grade by

$$T_{-i+1/2} = \alpha_i[\mu_+, T_{-i}] + \beta_i[\bar{\mu}_+, T_{-i}] \qquad \alpha_i \neq \beta_i. \quad (2.29)$$

Moreover, we require the sets $\{T_{-i}, [\mu_+, T_{-i}],\ 1 \leqslant i \leqslant n\}$ and $\{T_{-i}, [\bar{\mu}_+, T_{-i}],\ 1 \leqslant i \leqslant n\}$ to span abelian superalgebras. We shall exhibit later three choices for the space $V_{\leqslant 0}$ satisfying these requirements. Then it may be shown iteratively that equation (2.28), together with the nonlinear constraints (2.11) and (2.12) completely determine the gauge-fixed form of $(B_+)_{\leqslant 0}$ and $(\bar{B}_+)_{\leqslant 0}$ to be

$$(B_+)^{\mathrm{gf}}_{\leqslant 0} = \sum_{i=1}^{n} \alpha_i(-D_+W_{i+1/2}T_{-i} + W_{i+1/2}[\mu_+, T_{-i}]) \quad (2.30)$$

$$(\bar{B}_+)^{\mathrm{gf}}_{\leqslant 0} = \sum_{i=1}^{n} \beta_i(-\bar{D}_+W_{i+1/2}T_{-i} + W_{i+1/2}[\bar{\mu}_+, T_{-i}]). \quad (2.31)$$

From this we conclude that there are only n independent polynomials $W_{i+1/2}$, $1 \leqslant i \leqslant n$. This is half the number of conserved currents found in the $N = 1$ framework, which was to be expected since an unconstrained $(2, 0)$ superfield contains two $(1, 0)$ superfields. Let us give examples of gauges satisfying our requirements. The first was given in [2], it is a vertical gauge where $T_{-i} = E_{2i+1,1}$, $T_{-i+1/2} = E_{2i,1} = [\bar{\mu}_+, T_{-i}]$. Then the gauge-fixed forms are

$$(B_+)^{\mathrm{gf}}_{\leqslant 0} = 0 \qquad (\bar{B}_+)^{\mathrm{gf}}_{\leqslant 0} = \sum_{i=1}^{n}(-\bar{D}_+ V_i E_{2i+1,1} + V_i E_{2i,1}). \tag{2.32}$$

One may construct as well a horizontal gauge by choosing $T_{-n+i-1} = E_{2n+1,2i-1}$ and $T_{-n+i-1/2} = E_{2n+1,2i} = -[\mu_+, T_{-n+i-1}]$. The gauge-fixed forms are

$$(B_+)^{\mathrm{gf}}_{\leqslant 0} = \sum_{i=1}^{n}(D_+ X_i E_{2n+1,2i-1} + X_i E_{2n+1,2i}) \qquad (\bar{B}_+)^{\mathrm{gf}}_{\leqslant 0} = 0. \tag{2.33}$$

A third possibility is to take an $osp(2|1)$ lowest weight gauge, that is to say $T_{-i} = (E_{--})^i$ and $T_{-i+1/2} = [\mu_+ - \bar{\mu}_+, (E_{--})^i]$, where the matrix E_{--} has been defined in equation (1.10). Then both gauge-fixed forms are non-zero and read

$$(B_+)^{\mathrm{gf}}_{\leqslant 0} = \sum_{i=1}^{n}(-D_+ Z_i T_{-i} + Z_i[\mu_+, T_{-i}]) \tag{2.34}$$

$$(\bar{B}_+)^{\mathrm{gf}}_{\leqslant 0} = \sum_{i=1}^{n}(\bar{D}_+ Z_i T_{-i} - Z_i[\bar{\mu}_+, T_{-i}]). \tag{2.35}$$

It is in this gauge that the conserved currents have simple $N = 2$ superconformal transformations. From equations (2.15), (2.19) and (2.20) we find the superconformal transformations of B_+ and $\bar{B}_+$ to be

$$\delta B_+ = K(\epsilon^{++}, \epsilon^{--})B_+ + \tfrac{1}{2} D_+\bar{D}_+\epsilon^{++} B_+ + [l_0, B_+] + D_+ l_0 \tag{2.36}$$

$$\delta \bar{B}_+ = K(\epsilon^{++}, \epsilon^{--})\bar{B}_+ + \tfrac{1}{2} \bar{D}_+ D_+\epsilon^{++} \bar{B}_+ + [l_0, \bar{B}_+] + \bar{D}_+ l_0. \tag{2.37}$$

The lowest weight gauge (2.35) is not stable under these transformations. However, only a very simple compensating gauge transformation $1 + \delta G$ is needed, where δG is non-zero only at grades $-\frac{1}{2}$ and -1. One finds that all currents beside Z_1 are $N = 2$ superprimary fields,

$$\delta Z_j = K(\epsilon^{++}, \epsilon^{--})Z_j + j\partial_{++}\epsilon^{++} Z_j \tag{2.38}$$

and Z_1 has the transformation law of a super-energy–momentum tensor

$$\delta Z_1 = K(\epsilon^{++}, \epsilon^{--})Z_1 + \partial_{++}\epsilon^{++} Z_1 + \tfrac{1}{2}[D_+, \bar{D}_+]\partial_{++}\epsilon^{++}. \tag{2.39}$$

The $N = 2$ Miura transformation, expressing the conserved currents V_i in equation (2.32) in terms of the Toda superfield ϕ^i is discussed in [2]. We will meet this transformation again in the context of the $N = 2$ supersymmetric KdV equation in section 3.

2.4. Non-abelian N = 2 Toda equations

It is clear how the methods developed in [2] and in the present article can be generalized to the case of non-abelian Toda equations. We shall not try to construct a general theory here, but restrict our attention to the Toda equations associated with an $sl(2|1)$ embedding inside $sl(n+1|n)$. Such equations have the property that one can find a generating set for the conserved currents such that all elements beside the $N = 2$ superconformal tensor are $N = 2$ superprimary fields. The classification of $sl(2|1)$ embeddings inside a simple superalgebra has been established in [11]. In our case, it amounts to decompose the $2n + 1 \times 2n + 1$ matrices into blocks so that the size of the diagonal blocks is odd. These diagonal blocks form a regular subalgebra. The semisimple part of this subalgebra is a sum of superalgebras of the type $sl(n_i+1|n_i)$ or $sl(m_i|m_i+1)$. We then consider the $sl(2|1)$ embedding which is principal in this superalgebra. Thus μ_+ ($\bar{\mu}_+$) is still the sum of those matrices $E_{2i-1,2i}$ ($E_{2i,2i+1}$) which are inside the diagonal blocks. All this is very reminiscent of the construction of $sl(2)$ subalgebras of $sl(n)$. A difference is that in the case of $sl(2|1)$ embeddings, the ordering of the blocks is of some importance. For examples, the decompositions $5 = 3+1+1$ and $5 = 1+3+1$ lead to different embeddings. Just as in the principal case, the $osp(1|2)$ generator $e_0 = H + \bar{H}$ induces a half-integer gradation of the superalgebra $sl(n+1|n)$, and $\mathcal{G}_0$ is the subalgebra of grade zero. The Toda superfield $g_0(x,\theta,\bar{\theta})$ belongs to the corresponding supergroup G_0. We may then consider the connections

$$\tilde{B}_+ = \tilde{D}_+ g_0 g_0^{-1} + \tilde{\mu}_+ \qquad \tilde{B}_- = -\hat{g}_0 \tilde{\mu}_- g_0^{-1}. \tag{2.40}$$

Then we write for these connections the zero curvature equations (2.11), (2.12) and (2.13). The equations involving only one light-cone chirality lead to the constraints

$$\begin{aligned} &[\mu_+, D_+ g_0 g_0^{-1}\} = 0 \qquad [\bar{\mu}_+, \bar{D}_+ g_0 g_0^{-1}\} = 0 \\ &[\mu_-, \hat{g}_0^{-1} D_- g_0\} = 0 \qquad [\bar{\mu}_-, \hat{g}_0^{-1} \bar{D}_- g_0\} = 0. \end{aligned} \tag{2.41}$$

The equations (2.14) involving both light-cone chiralities lead to the four dynamical equations

$$\tilde{D}_-(\tilde{D}_+ g_0 g_0^{-1}) + \tilde{\mu}_+ \hat{g}_0 \tilde{\mu}_- g_0^{-1} + g_0 \tilde{\mu}_- \hat{g}_0^{-1} \tilde{\mu}_+ = 0. \tag{2.42}$$

This is not the end of the story, since one can easily check that, even taking into account the constraints (2.41), the superfield g_0 contains too many components. The situation here is the same as in the $N = 2$ WZNW model [10], and we expect the constraints on the supercurrents to be constructed from two classical r-matrices r_L and r_R of the superalgebra $\mathcal{G}_0$. An r-matrix $r = r_L$ or $r = r_R$ is a linear transformation of $\mathcal{G}_0$ satisfying the classical modified Yang–Baxter equation

$$r[rM, N\} + r[M, rN\} - [rM, rN\} = [M, N\}. \tag{2.43}$$

Moreover, we require r to be a super-antisymmetric involution

$$r^2 = Id \qquad \mathrm{str}(rM)N = -\,\mathrm{str}\, M(rN). \tag{2.44}$$

We note $\mathcal{G}_0^+$ and $\mathcal{G}_0^-$ the eigenspaces of r with respective eigenvalues 1 and -1. Then the properties of r show that $\mathcal{G}_0^+$ and $\mathcal{G}_0^-$ are isotropic subalgebras of $\mathcal{G}_0$. The data ($\mathcal{G}_0$, $\mathcal{G}_0^+$, $\mathcal{G}_0^-$) is called a Manin triple.

Let us call $\mathcal{G}_L^+$ and $\mathcal{G}_R^+$ the eigenspaces of r_L and r_R with eigenvalue 1, $\mathcal{G}_L^-$ and $\mathcal{G}_R^-$ the eigenspaces of r_L and r_R with eigenvalue -1. Then we expect the form of the constraints on the super-currents to be

$$\begin{aligned} &D_+ g_0 g_0^{-1} \in \mathcal{G}_R^+ \qquad \bar{D}_+ g_0 g_0^{-1} \in \mathcal{G}_R^- \\ &\hat{g}_0^{-1} D_- g_0 \in \mathcal{G}_L^+ \qquad \hat{g}_0^{-1} \bar{D}_- g_0 \in \mathcal{G}_L^-. \end{aligned} \tag{2.45}$$

The possible choices for the r-matrices are severely restricted if we require that the constraints (2.41) coming from the zero curvature equations should be a subset of the complete constraints (2.45). This turns out to be easily realized. We use the decompositions of the superalgebra $\mathcal{G}_0$ as

$$\mathcal{G}_0 = (\mathrm{Im\ ad}\mu_+)_0 \oplus (\mathrm{Im\ ad}\bar{\mu}_+)_0 \oplus \mathcal{G}_0^S = (\mathrm{Im\ ad}\mu_-)_0 \oplus (\mathrm{Im\ ad}\bar{\mu}_-)_0 \oplus \mathcal{G}_0^S \tag{2.46}$$

where $\mathcal{G}_0^S$ is the singlet part of $\mathcal{G}_0$ under the $sl(2|1)$ action. The set Im adμ_+ is easily shown to be a superalgebra on which the invariant quadratic form vanishes. Then we may take

$$(\text{Im ad}\mu_+)_0 \subset \mathcal{G}_R^+ \qquad (\text{Im ad}\bar{\mu}_+)_0 \subset \mathcal{G}_R^-$$
$$(\text{Im ad}\mu_-)_0 \subset \mathcal{G}_L^+ \qquad (\text{Im ad}\bar{\mu}_-)_0 \subset \mathcal{G}_L^-. \tag{2.47}$$

The subalgebra $\mathcal{G}_0^S$ is orthogonal to the spaces Im ad$\mu_\pm$, Im ad$\bar{\mu}_\pm$, and one has the commutation relations

$$[\mathcal{G}_0^S, \text{Im ad}\mu_\pm\} \in \text{Im ad}\mu_\pm \qquad [\mathcal{G}_0^S, \text{Im ad}\bar{\mu}_\pm\} \in \text{Im ad}\bar{\mu}_\pm. \tag{2.48}$$

From these equations we conclude that we may choose freely the definition of r_R and r_L inside $\mathcal{G}_0^S$. We may for instance take the upper (lower) triangular elements in $\mathcal{G}_0^S$ to be included in $\mathcal{G}_{R,L}^+$ ($\mathcal{G}_{R,L}^-$). We denote by $\mathbf{1}_i$ the identity matrix in the ith block. Then the matrices $H_i = \mathbf{1}_i + \mathbf{1}_{i+1}$ span the even-dimensional Cartan subalgebra of $\mathcal{G}_0^S$. We may take $H_{2i-1} \in \mathcal{G}_{R,L}^+$ and $H_{2i} \in \mathcal{G}_{R,L}^-$. With these choices, the complete constraints (2.45) imply the constraints (2.41).

The construction of a gauge invariant formulation of the $N = 2$ non-abelian Toda equations follows exactly the same line as the abelian case. The only difference is that beside the constraints (2.14), one should add gauge invariant constraints on the $sl(2|1)$ singlet part of B_+, $\bar{B}_+$, C_-, $\bar{C}_-$, of the form

$$(B_+)_S \in \mathcal{G}_R^+ \qquad (\bar{B}_+)_S \in \mathcal{G}_R^- \qquad (C_-)_S \in \mathcal{G}_L^+ \qquad (\bar{C}_-)_S \in \mathcal{G}_L^+. \tag{2.49}$$

3. $N = 2$ KdV equation in superspace

The methods used in the first section for the description of the $N = 2$ supersymmetric Toda equations in extended superspace apply as well to the $N = 2$ KdV equation. The formulation in $N = 2$ superspace that we shall give is strongly inspired from the $N = 1$ treatment given in [9]. Most of the notations that we use also come from [9].

3.1. Lax operators, gauge invariance

The relevant superalgebra is now the untwisted loop algebra constructed from $sl(2|2)$, or rather the quotient $A^{(1)}(1|1)$ of this algebra by its centre. We thus consider the set of the 4×4 real matrices depending on a loop parameter λ. As in the first section, the supertrace is defined as the alternate sum of diagonal elements, and we consider the superalgebra of matrices with zero supertrace. Any function of λ multiplying the identity matrix is in the centre of this algebra. We choose a representative in each equivalence class of the quotient by the centre by requiring the last element on the diagonal to vanish. We introduce the following odd elements

$$\omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \bar{\omega} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \end{pmatrix} \qquad \Lambda = \omega + \bar{\omega} \tag{3.1}$$

which satisfy

$$\omega^2 = 0 \qquad \bar{\omega}^2 = 0 \qquad \{\omega, \bar{\omega}\} = \Lambda^2. \tag{3.2}$$

Λ^2 is an even semisimple element of the superalgebra. We shall use the notations

$$\mathcal{K} = \ker \text{ad}\Lambda^2 \qquad \mathcal{K} = [\mathcal{K}, \mathcal{K}\} \oplus \mathcal{K}' \tag{3.3}$$

where $[\mathcal{K}, \mathcal{K}\}$ denotes the commutator subalgebra of $\mathcal{K}$, and $\mathcal{K}'$ some complement. We consider a gradation d of the superalgebra defined by $d(\lambda^p E_{i,j}) = 4p + j - i$. Notice that

ω and $\bar{\omega}$ have grade 1, and Λ^2 has grade 2. The $N=2$ superspace has a bosonic coordinate x and two Grassmann coordinates θ, $\bar{\theta}$. Supersymmetric covariant derivatives are

$$D = \frac{\partial}{\partial\theta} + \bar{\theta}\partial \qquad \bar{D} = \frac{\partial}{\partial\bar{\theta}} + \theta\partial \qquad \partial = \frac{\partial}{\partial x}. \tag{3.4}$$

In analogy with the methods developed in [8], we introduce the spinor Lax operators

$$\mathcal{L} = D + q + \omega \qquad \bar{\mathcal{L}} = \bar{D} + \bar{q} + \bar{\omega} \tag{3.5}$$

where the superfields q and $\bar{q}$ are λ-independent odd matrices of non-positive grade. In particular, the upper triangular elements of q and $\bar{q}$ vanish. The operators $\mathcal{L}$ and $\bar{\mathcal{L}}$ are required to satisfy zero curvature equations. Two of these equations are constraints on q and $\bar{q}$

$$\hat{\mathcal{L}}\mathcal{L} = 0 \Rightarrow Dq + \hat{q}q + [\hat{q}, \omega\} = 0 \tag{3.6}$$

$$\hat{\bar{\mathcal{L}}}\bar{\mathcal{L}} = 0 \Rightarrow \bar{D}\bar{q} + \hat{\bar{q}}\bar{q} + [\hat{\bar{q}}, \bar{\omega}\} = 0 \tag{3.7}$$

while the third simply determines the vector Lax operator

$$\mathcal{L}_x = \hat{\mathcal{L}}\bar{\mathcal{L}} + \hat{\bar{\mathcal{L}}}\mathcal{L} = 2\partial + Q - \Lambda^2. \tag{3.8}$$

As in the Toda case, equations (3.6) and (3.7) may be interpreted as follows. adω being a nilpotent element of the superalgebra, its image is included in its Kernel. The elements in q which do not belong to the Kernel are fully determined in terms of those in the Kernel. Moreover, the elements in q belonging to the image of adω are unconstrained $N=2$ superfields, while those belonging to the Kernel but not to the image satisfy chirality-type constraints. Similar statements hold for $\bar{q}$ and $\bar{\omega}$.

The form (3.5) of the Lax operators and the zero curvature equations (3.6) and (3.7) are invariant under the gauge transformations

$$\mathcal{L} \to \hat{g}\mathcal{L}g^{-1} \qquad \bar{\mathcal{L}} \to \hat{g}\bar{\mathcal{L}}g^{-1} \tag{3.9}$$

where $g(x, \theta, \bar{\theta})$ is a λ-independent lower triangular matrix with 1's along the diagonal. Two particular gauge-fixings will be useful in the sequel. One is a vertical gauge

$$q^1_{gf} = 0 \qquad \bar{q}^1_{gf} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \bar{D}W & 0 & 0 & 0 \\ W & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.10}$$

where W is an unconstrained $N=2$ superfield. As in the Toda case, the existence of this gauge is shown by using the $N=1$ results established in [9] and taking into account the constraints (3.6) and (3.7). Notice that this gauge is not of Drinfeld–Sokolov type. That is to say that the gauge group element which brings q and $\bar{q}$ to this form is not a local differential polynomial in the matrix elements of q and $\bar{q}$. The other gauge we shall consider is a diagonal gauge

$$q^2_{gf} = \begin{pmatrix} \phi & 0 & 0 & 0 \\ 0 & \phi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \bar{q}^2_{gf} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \bar{\phi} & 0 & 0 \\ 0 & 0 & \bar{\phi} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.11}$$

where ϕ is chiral, $D\phi = 0$, and $\bar{\phi}$ is anti-chiral, $\bar{D}\bar{\phi} = 0$. Although the gauge (3.10) is not of Drinfeld–Sokolov type, the superfield W may be expressed as a differential polynomial in terms of the superfields ϕ and $\bar{\phi}$. This is the celebrated Miura transformation, which is most easily obtained by introducing a 4-vector $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^t$ which is annihilated by the Lax operators

$$\mathcal{L}\psi = 0 \qquad \bar{\mathcal{L}}\psi = 0. \tag{3.12}$$

The first component ψ_1 of the vector ψ is gauge invariant. The matrix equations (3.12) reduce to scalar gauge-invariant differential equations on ψ_1. In the vertical gauge, one obtains

$$\bar{D}\psi_1 = 0 \qquad \bar{D}D(\partial\psi_1 - W\psi_1) = \lambda\psi_1 \tag{3.13}$$

and in the diagonal gauge

$$\bar{D}\psi_1 = 0 \qquad \bar{D}D(\bar{D} + \bar{\phi})(D + \phi)\psi_1 = \lambda\psi_1 \tag{3.14}$$

which leads to the relation $W = -(\phi\bar{\phi} + \bar{D}\phi + D\bar{\phi})$.

3.2. *Evolution equations, conserved charges*

We wish to write evolution equations for the Lax operators $\mathcal{L}$ and $\bar{\mathcal{L}}$ of the type

$$\frac{\partial\mathcal{L}}{\partial t} = A\mathcal{L} - \mathcal{L}\hat{A} \qquad \frac{\partial\bar{\mathcal{L}}}{\partial t} = A\bar{\mathcal{L}} - \bar{\mathcal{L}}\hat{A} \tag{3.15}$$

and A is an even matrix belonging to the superalgebra $A^{(1)}(1|1)$. As in the bosonic case [8], we first need to construct an even matrix M commuting with $\mathcal{L}$ and $\bar{\mathcal{L}}$

$$M\mathcal{L} - \mathcal{L}\hat{M} = 0 \qquad M\bar{\mathcal{L}} - \bar{\mathcal{L}}\hat{M} = 0. \tag{3.16}$$

We decompose M as $M = M^+ + M^-$, where the grades in M^+ are positive or zero, and the grades in M^- are strictly negative. Then we can take $A = M^+$. In order to construct the matrix M, we shall show that there exists a matrix $F = \sum_{n=1}^{\infty} F_{-n}$, $d(F_{-n}) = -n$, which brings $\mathcal{L}$ and $\bar{\mathcal{L}}$ to the form

$$\mathrm{e}^{\hat{F}}\mathcal{L}\,\mathrm{e}^{-F} = D + H + \omega \qquad \mathrm{e}^{\hat{F}}\bar{\mathcal{L}}\,\mathrm{e}^{-F} = \bar{D} + \bar{H} + \bar{\omega} \tag{3.17}$$

where $H = \sum_{n=1}^{\infty} H_{-n}$ and $\bar{H} = \sum_{n=1}^{\infty} \bar{H}_{-n}$ belong to the Kernel $\mathcal{K}$ of adΛ^2. At any finite grade, F, H and $\bar{H}$ depend polynomially on the matrix elements of q and $\bar{q}$, and on their derivatives. Suppose that the proof has been carried out down to the grade $-n+1$ for H and $\bar{H}$, and down to the grade $-n$ for F. At the next grade, it will be convenient to look first at the sum $H + \bar{H}$. We have

$$H_{-n} + \bar{H}_{-n} = P(H_{-p}, \bar{H}_{-p}, F_{-p}) + [\hat{F}_{-n-1}, \Lambda\} \tag{3.18}$$

where P is a differential polynomial in H_{-p}, $\bar{H}_{-p}$ with $p < n$ and in F_{-p} with $p \leqslant n$. We choose the particular form $F_{-n-1} = \hat{G}_{-n-2}\Lambda - \Lambda G_{-n-2}$. Then the last equation becomes

$$H_{-n} + \bar{H}_{-n} = P(H_{-p}, \bar{H}_{-p}, F_{-p}) + [\Lambda^2, G_{-n-2}]. \tag{3.19}$$

We then use the fact that Λ^2 is semisimple to conclude that we may choose G_{-n-2} in such a way that $H_{-n} + \bar{H}_{-n}$ belongs to $\mathcal{K}$. To show that both H_{-n} and $\bar{H}_{-n}$ belong to $\mathcal{K}$, we use the constraints (3.6), (3.7) coming from the zero curvature equations. They lead at grade $-n+1$ to the equations

$$DH_{-n+1} + \sum_{p+q=n-1} [\hat{H}_{-p}, H_{-q}\} + [\hat{H}_{-n}, \omega\} = 0 \tag{3.20}$$

$$\bar{D}\bar{H}_{-n+1} + \sum_{p+q=n-1} [\hat{\bar{H}}_{-p}, \bar{H}_{-q}\} + [\hat{\bar{H}}_{-n}, \bar{\omega}\} = 0 \tag{3.21}$$

$$\Rightarrow [\hat{H}_{-n}, \omega\} \in \mathcal{K} \qquad [\hat{\bar{H}}_{-n}, \bar{\omega}\} \in \mathcal{K}. \tag{3.22}$$

Thus we know that $H_{-n}+\bar{H}_{-n}$, $[\hat{H}_{-n},\omega\}$ and $[\hat{\bar{H}}_{-n},\bar{\omega}\}$ are in $\mathcal{K}$. Taking a supercommutator of $H_{-n}+\bar{H}_{-n}$ with ω or $\bar{\omega}$ shows that also $[\hat{H}_{-n},\bar{\omega}\}$ and $[\hat{\bar{H}}_{-n},\omega\}$ are in $\mathcal{K}$. Then, using the fact that $\Lambda^2=\{\omega,\bar{\omega}\}$ is semisimple, we conclude that H_{-n} and $\bar{H}_{-n}$ belong to $\mathcal{K}$. It will be useful to split H and $\bar{H}$ as

$$H=C+H' \qquad \bar{H}=\bar{C}+\bar{H}' \qquad C,\bar{C}\in[\mathcal{K},\mathcal{K}\} \qquad H',\bar{H}'\in\mathcal{K}'. \tag{3.23}$$

On H' and $\bar{H}'$, the constraints (3.20), (3.21) reduce to the chirality conditions $DH'=0$, $\bar{D}\bar{H}'=0$. Notice also that e^F is only defined up to a left multiplication by e^T, with T in $\mathcal{K}$. As a consequence, H' and $\bar{H}'$ are only defined up to the addition of a total derivative

$$H'\to H'+DT' \qquad \bar{H}'\to\bar{H}'+\bar{D}T'. \tag{3.24}$$

Then, the integrated quantities

$$Q=\int \mathrm{d}V_c H'=\int \mathrm{d}x\,(\bar{D}H')_{\theta=\bar{\theta}=0} \tag{3.25}$$

$$\bar{Q}=\int \mathrm{d}V_a H'=\int \mathrm{d}x\,(D\bar{H}')_{\theta=\bar{\theta}=0} \tag{3.26}$$

are not uniquely defined

$$Q\to Q+\int \mathrm{d}x\,(\bar{D}DT')_{\theta=\bar{\theta}=0} \qquad \bar{Q}\to\bar{Q}+\int \mathrm{d}x\,(D\bar{D}T')_{\theta=\bar{\theta}=0}. \tag{3.27}$$

However, the sum $Q+\bar{Q}$ is uniquely defined

$$Q+\bar{Q}\to Q+\bar{Q}+\int \mathrm{d}x\,(2\partial T')_{\theta=\bar{\theta}=0}=Q+\bar{Q}. \tag{3.28}$$

The fact that this quantity is uniquely defined also implies that it is a gauge invariant functional of q and $\bar{q}$. As we shall see next, $Q_{-n}+\bar{Q}_{-n}$ are the conserved charges of the $N=2$ KdV hierarchy. From now on, things work as in the bosonic case. We introduce the matrix

$$M=\mathrm{e}^{-\hat{F}}b\,\mathrm{e}^{\hat{F}} \tag{3.29}$$

where b is a constant matrix in the centre of $\mathcal{K}$. Then we choose

$$A=M^+=(\mathrm{e}^{-\hat{F}}b\,\mathrm{e}^{\hat{F}})^+. \tag{3.30}$$

The equations (3.15) should be seen as evolution equations for gauge invariant differential polynomials of q and $\bar{q}$. Let us study these equations on H and H'. Using the notation

$$B=\mathrm{e}^{\hat{F}}\frac{\partial}{\partial t}\mathrm{e}^{-\hat{F}}+\mathrm{e}^{\hat{F}}A\,\mathrm{e}^{-\hat{F}} \tag{3.31}$$

we obtain

$$\frac{\partial H}{\partial t}+DB=B(H+\omega)-(H+\omega)\hat{B} \tag{3.32}$$

$$\frac{\partial\bar{H}}{\partial t}+\bar{D}B=B(\bar{H}+\bar{\omega})-(\bar{H}+\bar{\omega})\hat{B}. \tag{3.33}$$

It is a consequence of these equations that B belongs to $\mathcal{K}$. Then the right-hand side of these equations belongs to $[\mathcal{K},\mathcal{K}\}$, and if we restrict to $\mathcal{K}'$ we find

$$\frac{\partial H'}{\partial t}+DB'=0 \qquad \frac{\partial\bar{H}'}{\partial t}+\bar{D}B'=0 \tag{3.34}$$

which implies that $Q+\bar{Q}$ is time independent.

We studied in some detail the case $b = \lambda\Lambda^2$, which has grade 6. We computed the evolution equation for the superfield W appearing in the gauge choice (3.10). When computed on some gauge-fixed form of the Lax operators, the evolution equations (3.15) acquire an extra term

$$\frac{\partial \mathcal{L}_{gf}}{\partial t} = (A+R)\mathcal{L}_{gf} - \mathcal{L}_{gf}(\hat{A}+\hat{R}) \qquad \frac{\partial \bar{\mathcal{L}}_{gf}}{\partial t} = (A+R)\bar{\mathcal{L}}_{gf} - \bar{\mathcal{L}}_{gf}(\hat{A}+\hat{R}) \tag{3.35}$$

where R belongs to the gauge algebra. The computation of the objects A, F and R is somewhat lengthy. The final result is

$$\partial W/\partial t = 2\partial^3 W - \tfrac{3}{2}\partial(DW\bar{D}W) - \tfrac{1}{4}\partial W^3. \tag{3.36}$$

This $N = 2$ KdV equation was obtained in [12]. Extensions of these results to the KdV hierarchies associated with the superalgebras $sl(n|n)$ is in principle straightforward.

4. Conclusion

Let us consider the relation between our gauge invariant equations and the WZNW models. In the bosonic [3] or $N = 1$ [4] cases, the gauge invariant field equations may be considered as the field equations of a gauged WZNW model. This means in particular that if the gauge fields are set to zero, and the field equations coming from the variation of these gauge fields are dropped, one recovers the equation of motion of the WZNW model. In our case, since we do not have an action, it is not clear which are the equations coming from the variation of the gauge superfields. Naively, and in analogy with the bosonic case, one could expect that the field equations of the connections are the constraints (2.14). If these constraints are dropped, and the gauge fields are set to zero, one ends up with the following equations

$$\tilde{D}_-(\tilde{D}_+ g g^{-1}) = 0. \tag{4.1}$$

These are very nice equations, possessing a very big loop invariance, but they are not the $N = 2$ WZNW equations. In particular, the $N = 2$ unconstrained superfield g contains a dynamical vector field. The true $N = 2$ WZNW equations [10] contain equation (4.1), together with constraints on the superfield g constructed from two classical r-matrices. These constraints cannot be considered as a subset of the gauge-invariant constraints (2.14). Thus at present the relation of our gauge invariant formulation with a gauged WZNW model is unclear.

It is not a very big surprise to verify that methods which work in ordinary space and in $N = 1$ superspace do extend to $N = 2$ superspace at the level of field equations. It is considerably more difficult to obtain in extended superspace actions or Hamiltonian formulations. However, as already stressed, an Hamiltonian reduction approach to $N = 2$ $\mathcal{W}$ algebras has been developed in [6]. This approach makes use of the $N = 2$ operator product expansions for constrained supercurrents constructed in [10]. These operator product expansions, or rather the Poisson brackets which may be derived from them, would be relevant in a Hamiltonian formulation of the non-abelian $N = 2$ Toda equations discussed at the end of section 2. It is however not clear to us how they could be used in the gauge-invariant approach to Toda or KdV equations.

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